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Negative Acceleration Components for a Relativistic Particle

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In relativistic mechanics it is of great importance to distinguish between those properties of a physical system that are frame independent or intrinsic to it and those properties that are contrived or that depend for their occurrence on observing the system from a particular reference frame. In recent years there has been renewed interest in the three-vector Newtonian acceleration

$$\mathbf{a}(t) = \frac{d\mathbf{v}}{dt} \tag{1}$$

of a particle acted upon by an external force, and under what circumstances various components of $\mathbf{a}$ are opposite in direction to the applied force. Recent examinations of the Newtonian acceleration under an external force have been limited to very special cases such as for a two-dimensional[1] or three-dimensional[2] motion at the instant for which $v_x(t) = v_y(t)$ or $v_x(t) = v_y(t) = v_z(t)$, or for the case when the force is time independent.[3] It is our purpose: (i) to demonstrate that a number of the conclusions arrived at numerically for the above special cases are more generally true for an arbitrary time-dependent applied force $\mathbf{F}(t)$ and (ii) to point out which of these properties of the Newtonian acceleration are rotationally invariant and which properties depend for their manifestation on being in a particular reference frame. Specifically we find that the component of $\mathbf{a}(t)$ in the direction of $\mathbf{F}(t)$ is always positive, implying that for a given coordinate system, all three components of $\mathbf{a}(t)$ can never be opposite in sign to the respective components of $\mathbf{F}(t)$. However, whether zero, one, or two of the acceleration components are directed opposite to the force components depends on which reference frame one is observing the particle from.

We consider a relativistic particle of rest mass m_0 with initial three-momentum $\mathbf{p}_0$ acted upon by the arbitrary time-dependent external force $\mathbf{F}(t)$. Integrating Newton's second law

$$d\mathbf{p} = \mathbf{F}(t')dt' \tag{2}$$

from the initial time to t yields

$$\mathbf{p}(t) = \frac{m_0\mathbf{v}(t)}{(1 - v^2/c^2)^{1/2}} = \mathbf{p}_0 + \int_0^t \mathbf{F}(t')dt' = \mathbf{p}_0 + \mathbf{I}(t) \tag{3}$$

or

$$\mathbf{v}(t) = \frac{\mathbf{p}_0 + \mathbf{I}(t)}{\{m_0^2 + [\mathbf{p}_0 + \mathbf{I}(t)]^2/c^2\}^{1/2}}. \tag{4}$$

The Newtonian acceleration of Eq. (1) is then

$$\mathbf{a}(t) = \frac{[m_0^2 + B^2(t)]\,\mathbf{F}(t) - [\mathbf{B}(t)\cdot\mathbf{F}(t)]\mathbf{B}(t)}{[m_0^2 + B^2(t)]^{3/2}} \tag{5}$$

with

$$\mathbf{B}(t) \equiv [\mathbf{p}_0 + \mathbf{I}(t)]/c = \mathbf{p}(t)/c. \tag{6}$$

The component of $\mathbf{a}$ in the direction of $\mathbf{F}$ is

$$a_F = \frac{\mathbf{a}(t)\cdot\mathbf{F}(t)}{|\mathbf{F}(t)|} = \frac{m_0^2F^2(t) + B^2(t)F^2(t) - [\mathbf{B}(t)\cdot\mathbf{F}(t)]^2}{|\mathbf{F}(t)|\,[m_0^2 + B^2(t)]^{3/2}} \geq 0, \tag{7}$$

which is nonnegative for all times. Imposing a three-dimensional coordinate system on the force and acceleration vectors we have

$$\mathbf{a}\cdot\mathbf{F} = a_xF_x + a_yF_y + a_zF_z \geq 0. \tag{8}$$

If all three components of the acceleration were opposite in sign to the components of the force, the left-hand side of Eq. (8) would be negative, which is disallowed by Eq. (7). Thus as a consequence of $\mathbf{a}\cdot\mathbf{F} \geq 0$, all three components of $\mathbf{a}$, independent of reference frame, cannot be directed opposite to the components of $\mathbf{F}$.

To demonstrate that whether zero, one, or two components of $\mathbf{a}$ are directed opposite to those of $\mathbf{F}$ depends on one's reference frame, we consider the initial frame in which $\mathbf{F}$ is directed along the positive z axis and $\mathbf{a}$ is in the xz plane with the components

$$\mathbf{a} = a\sin\Omega\,\mathbf{u}_x + a\cos\Omega\,\mathbf{u}_z, \tag{9}$$

with

$$\cos\Omega \equiv \mathbf{a}\cdot\mathbf{F}/|\mathbf{a}||\mathbf{F}| \tag{10}$$

and

$$0 \leq \Omega \leq \pi/2. \tag{11}$$

Rotating the vectors $\mathbf{F}$ and $\mathbf{a}$ via the Euler angles, i.e., a rotation of $-\gamma$ about the z axis, then a rotation of $-\theta$ with $0 \leq \theta \leq \pi/2$ about the y axis, then a rotation of $-\phi$ with $0 \leq \phi \leq \pi/2$ about the z axis,[4] we obtain

$$\mathbf{F} = F\sin\theta\cos\phi\,\mathbf{u}_x + F\sin\theta\sin\phi\,\mathbf{u}_y + F\cos\theta\,\mathbf{u}_z \tag{12}$$

with all three components positive and

$$\begin{aligned}\mathbf{a} = {} & a[(\cos\gamma\cos\theta\cos\phi - \sin\gamma\sin\phi)\sin\Omega \\ & + \sin\theta\cos\phi\cos\Omega]\mathbf{u}_x \\ & + a[(\cos\gamma\cos\theta\sin\phi + \sin\gamma\cos\phi)\sin\Omega \\ & + \sin\theta\sin\phi\cos\Omega]\mathbf{u}_y \\ & + a[\cos\theta\cos\Omega - \cos\gamma\sin\theta\sin\Omega]\mathbf{u}_z.\end{aligned} \tag{13}$$

The choice, for example, of $\gamma = 0$ and $\theta < \pi/2 - \Omega$ gives

$$\mathbf{a} = a\cos\phi\sin(\theta + \Omega)\mathbf{u}_x + a\sin\phi\sin(\theta + \Omega)\mathbf{u}_y + a\cos(\theta + \Omega)\mathbf{u}_z \tag{14}$$

with all three components positive, while $\theta > \pi/2 - \Omega$ gives positive x and y components and a negative z component. On the other hand, $\gamma = \pi$ and $\theta < \Omega$ gives

$$\mathbf{a} = a\cos\phi\sin(\theta - \Omega)\mathbf{u}_x + a\sin\theta\sin(\theta - \Omega)\mathbf{u}_y + a\cos(\theta - \Omega)\mathbf{u}_z \tag{15}$$

with negative x and y components and a positive z component, thus demonstrating the claim that whether zero, one, or two components are negative depends on the choice of reference frame.

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